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Testing Viability of Hubble Parametrizations in *f(R,T)* Kaluza-Klein Gravity: A Stability Analysis

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Abstract

While various parametrizations of the Hubble parameter have been proposed to model the universe's expansion, their viability within higher-dimensional modified gravity remains largely untested. This study rigorously investigates the cosmological dynamics of five distinct Hubble models (hyperbolic tangent, logarithmic, power-law, exponential decay, and emergent) within a Kaluza-Klein *f(R,T)* gravity framework. Crucially, beyond standard kinematic evolution, we subject these models to a comprehensive stability analysis using the squared speed of sound and energy condition constraints. We demonstrate that while most parametrizations can successfully mimic late-time cosmic acceleration, only the Logamediate Model (Model III) maintains perturbative stability throughout the cosmic history. This work establishes a critical selection criterion for modified gravity reconstructions, filtering out observationally consistent but theoretically unstable dark energy candidates.

Keywords: FRW type Kaluza Klein Universe, Hubble Parameters, Modified Gravity, Stability, Energy conditions

1. Introduction

The accelerated expansion of the universe, as evidenced by observations of Type Ia supernovae, cosmic microwave background (CMB) anisotropies, and baryon acoustic oscillations (BAO), has posed significant challenges to the standard cosmological model. While the ΛCDM model, incorporating a cosmological constant (Λ) and cold dark matter (CDM), has been successful in explaining a wide range of cosmological observations, it faces theoretical issues such as the fine tuning and coincidence problems. These challenges have motivated the exploration of alternative theories of gravity that can account for cosmic acceleration without invoking dark energy. One such alternative is the *f(R,T)* gravity theory, proposed by Harko et al.,[1] which extends the standard *f(R)* gravity by including a dependence on the trace T of the energy-momentum tensor. This additional dependence allows for a more general coupling between matter and geometry, leading to modified field equations that can potentially explain the accelerated expansion of the universe through purely geometrical means. Subsequent studies have explored various aspects of *f(R,T)* gravity, including cosmological solutions,[2] thermodynamics,[3] and energy conditions.[4]

In parallel, higher-dimensional theories, such as the Kaluza-Klein (KK) theory, have been explored to unify gravity with other fundamental interactions. The KK framework introduces extra spatial dimensions, compactified to scales beyond current experimental detection, which can influence the dynamics of the four-dimensional universe. Incorporating KK geometry into cosmological models provides a richer structure to study the evolution of the universe and the effects of extra dimensions on observable quantities. Notable works in this area include the application of KK theory to cosmology,[5] the study of bulk metrics,[6] and the exploration of Lovelock-Cartan theory in higher dimensions.[7] Parametrizing the Hubble parameter $H(t)$ with specific functional forms has been a fruitful approach in cosmology, allowing for the reconstruction of the expansion history of the universe and the analysis of different evolutionary scenarios. Various parametrizations, such as power-law, exponential, and hyperbolic functions, have been employed to model different phases of cosmic evolution, including inflation, deceleration, and late-time acceleration.[8-10] Recent studies have proposed novel parametrization approaches to better fit observational data and explore the dynamics of dark energy.[11,12]

In this study, we investigate five different parametrizations of the Hubble parameter within the framework of *f(R,T)* gravity, considering a Kaluza-Klein-type Friedmann-Robertson-Walker (FRW) universe. We analyse the cosmological implications of each model, focusing on the evolution of key parameters such as the scale factor, deceleration parameter, energy density, pressure, and equation of state. Additionally, we examine the stability of the models and the validity of energy conditions, considering different spatial curvatures k=0 , ±1. Our aim is to provide a comprehensive analysis of how different Hubble parameter parametrizations affect the dynamics of the universe in the context of *f(R,T)* gravity with KK geometry. By comparing the results across different models and curvature scenarios, we seek to identify which models are physically viable. Crucially, beyond just matching the expansion history, we will test their theoretical consistency by performing a rigorous stability analysis and examining the standard energy conditions.

This paper is organized as follows: Section 2 details the theoretical framework, introducing the *f(R,T)* gravity field equations and the Kaluza-Klein FRW metric. Section 3 introduces the five distinct Hubble parameter models. For each model, we derive and analyse the key cosmological parameters, including the scale factor, deceleration parameter, energy density, pressure, and equation of state. Section 4 presents a comparative analysis of the kinematic results from all five models. Section 5 is dedicated to a rigorous stability analysis of each model by examining the squared speed of sound. Section 6 assesses the physical viability of the models by testing them against the standard energy conditions. Section 7 summarizes our findings and conclusions, highlighting the most physically consistent models.

2. Overview of f (R,T) Gravity, Metric and Field equations

For the *f(R,T)* modified theory of gravity formulated by,[1] the action for the modified theories of gravity takes the following form

$$S = \frac{1}{16\pi}\int f(R,T)\sqrt{-g}\,d^4x + \int L_m\sqrt{-g}\,d^4x\,, \tag{1}$$

where *f(R,T)* is an arbitrary function of the Ricci scalar, R and of the trace T of the stress-energy tensor of the matter $T\mu\nu$. Lm is the matter Lagrangian density.

In the literature given by,[1] varying the action (1) with respect to metric tensor $g_{\mu\nu}$ for the function *f(R,T)* is given by $f(R,T) = R+2f(T)$, where $f(T)$ is an arbitrary function of the trace of the stress-energy tensor of matter $T_{\mu\nu} = (p+\rho)u_\mu u_\nu - pg_{\mu\nu}$, where ρ and p being the density and pressure, respectively and u_μ is the four velocity satisfies the conditions $u_\mu u^\mu = 1$ and $u_\mu u^\mu = 0$, required gravitational field equations are given by

$$R_{\mu\nu} - \frac{1}{2}R\,g_{\mu\nu} = 8\pi T_{\mu\nu} + 2\,f'(T)T_{\mu\nu} + [2p\,f'(T) + f(T)]g_{\mu\nu}\,. \tag{2}$$

We consider a five-dimensional Kaluza-Klein space-time[41] where the extra spatial dimension is compactified.

$$ds^2 = dt^2 - a^2\left\{\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + sin^2\theta d\varphi^2) + (1-kr^2)d\phi^2\right\}, \tag{3}$$

where $a(t)$ is the scale factor of the universe, $k= -1,\ 0,\ +1$ is the curvature parameter for open, flat, closed models respectively.

Following the standard dimensional reduction procedure, we assume that the physical quantities depend only on the cosmic time t and not on the extra coordinate. The energy-momentum tensor $T_{\mu\nu}$ represents the effective matter distribution on the four-dimensional hypersurface as,

$$T_{\mu\nu} = (p+\rho)u_\mu u_\nu - pg_{\mu\nu}, \tag{4}$$

where u_μ is the five-velocity vector of the fluid which has components (1,0,0,0,0) satisfies $g^{ij}u_i u_j = -1$.

Thus, the quantities, $\rho(t)$ and $p(t)$ derived in this study correspond to the effective energy density and pressure observed in the 4D universe, incorporating the geometric contributions from the extra dimension and the *f(R,T)* coupling.

The field equations (2) with the choice of $f(T)=\lambda T$, λ is constant and using equation (4) for the metric (3) be written as,

$$\rho\lambda - p(8\pi + 4\lambda) = 3\dot{H} + 6H^2 + \frac{3\kappa}{a^2} \tag{5}$$

$$\rho(8\pi + 3\lambda) - 2p\lambda = 6H^2 + \frac{6k}{a^2}, \tag{6}$$

where overhead dot denotes differentiation with respect to t.

To solve the system of field equations (5) and (6), we adopt a reconstruction approach. In this framework, the cosmological dynamics are driven by the phenomenological choice of the Hubble parameter $H(t)$. Since the metric evolution is fully determined by $H(t)$, equations (5) and (6) constitute a system of two equations with two unknowns, $\rho(t)$ and $p(t)$. Unlike standard forward-modelling where an equation of state $p = \omega\rho$ is assumed a priori, here we determine the necessary fluid properties required to sustain the chosen expansion history. Consequently, the effective equation of state parameter is derived dynamically as:

$$\omega_{eff}(t) = p(t)/\rho(t) \tag{7}$$

This allows us to analyse how the fluid behaviour evolves from early-time deceleration to late-time acceleration without imposing a rigid barotropic constraint.

From equations (5), (6) and (7), we write, energy density, pressure and equation of state for KK FRW type universe in $f(R,T)$ gravity are as follows:

$$\rho(t) = \frac{Ah_2 - 2\lambda h_1}{AB - 2\lambda^2} \tag{8}$$

$$p(t) = \frac{\lambda h_2 - Bh_1}{AB - 2\lambda^2} \tag{9}$$

$$\omega(t) = \frac{\lambda h_2 - Bh_1}{Ah_2 - 2\lambda h_1} \tag{10}$$

where, $h_1 = 3\dot{H} + 6H^2 + \frac{3k}{a^2}$, $h_2 = 6H^2 + \frac{6k}{a^2}$, $A = (8\pi + 4\lambda)$ and $B = (8\pi + 3\lambda)$.

3. Role of Hubble Parameter in Cosmological Dynamics

The Hubble parameter $H(t)=(\dot{a}/a)$ plays a central role in cosmology, defining the rate of expansion of the universe and linking the dynamics of spacetime to its energy content through the Friedmann equations. Its evolution governs whether the universe is undergoing acceleration or deceleration, as quantified by the deceleration parameter $q(t)= -1- \dot{H}/H^2$. The specific form of $H(t)$ dictates the behaviour of key cosmological variables such as the energy density $\rho(t)$, pressure $p(t)$, the equation of state $\omega(t)$, and the scale factor $a(t)$. Accurate modelling of $H(t)$ is essential for describing different phases of cosmic evolution, including inflation, matter radiation domination, and late-time acceleration. Since the Hubble parameter is also directly constrained by observations through redshift measurements and cosmic chronometers, it serves as a critical bridge between theory and observation for testing various cosmological and modified gravity models.[16,18]

For the graphical analysis presented in this study, the model parameters (such as α, γ, β, δ, ψ, ξ, μ, n and λ) have been chosen strategically to illustrate the distinct physical behaviours of each cosmological model. The specific values are selected not as a best fit to observational data, but to clearly demonstrate the qualitative nature of the solutions. This includes showcasing key cosmic features, such as a potential transition from a decelerated phase ($q > 0$) to an accelerated one ($q < 0$), and to examine the physical viability of key parameters within the $f(R,T)$ Kaluza-Klein framework.

1. **Emergent exponential model:** $H(t) = [\gamma n e^{\gamma t}/(e^{\gamma t} + \mu)]$, a smooth function ideal for unified evolution from early to late times, where γ acts as an exponential rate.[29]
2. **Intermediate generalized power-law model:** $H(t)=\alpha\sigma t^{(\sigma-1)}$, representing early accelerated expansion phases, where σ is a power-law index.
3. **Logamediate model:** $H(t) = (\delta\beta \log^{\beta-1} t)/t$, $\delta>0$ and $\beta>1$.,which describes intermediate expansion behaviour, slower than de Sitter but faster than power-law, and is useful for modelling late-time acceleration.[13]
4. **Exponential decay model:** $H(t) = H_0 - \frac{H_1}{e^{\xi t}}$ capturing late-time deviations from ΛCDM and useful for testing dark energy dynamics, where ξ is an exponential decay rate.
5. **Emergent tanh expansion model:** $H(t) = \psi \tanh\left(\frac{t}{t_0}\right)$, allowing for a smooth transition from deceleration to acceleration, where the constant value ψ describes a late-time de Sitter exponential expansion.

3.1 Model I: Emergent exponential model

The Hubble parameter

$$H(t) = \frac{\gamma n e^{\gamma t}}{(e^{\gamma t} + \mu)} \tag{11}$$

represents a smoothly evolving cosmological expansion model characterized by a finite early-time value and asymptotic approach toward a constant expansion rate at late times. This form is particularly useful in modelling a transition from decelerated to accelerated expansion, which is a key feature of the observed cosmic evolution supported by Type Ia supernovae and cosmic microwave background data.[16] At early times, the function yields a finite expansion rate $H(t) \approx \frac{\gamma n}{(1+\mu)}$ thereby avoiding singularities and capturing a non-inflationary initial phase. As, $t \to \infty$ the Hubble rate approaches a constant value, $H(t) \to \gamma n$, effectively mimicking a de Sitter phase driven by a cosmological constant or dark energy. The deceleration parameter transitions from positive to negative values, indicating a natural evolution from deceleration to acceleration, in alignment with the standard cosmological timeline. Such a Hubble parameter is useful in the context of modified gravity theories and scalar field cosmology, as it allows for analytical reconstruction of potentials and matter interactions. It can also support emergent universe scenarios that do not originate from a Big Bang singularity, depending on parameter choices.[20,29,34] The model is flexible enough to accommodate both matter-dominated behaviour at intermediate times and dark-energy domination at late times, offering a unified description of cosmic history within a single analytical framework.

For analysing the evolution and stability within the higher-dimensional Kaluza-Klein framework, one can easily derived the scale factor and deceleration parameter from equation (11) as follows;

$$a(t) = a_0\,(e^{\gamma t} + \mu)^n \tag{12}$$

$$q(t) = -1 - \frac{\mu}{n e^{\gamma t}} \tag{13}$$

Furthermore, the expressions for energy density, pressure and equation of state parameter with the use of equations (11), (12) into (8)-(10) as follows;

$$\rho(t) = \frac{1}{D}\left[\frac{(8\pi+4\lambda)\,6\gamma^2 n^2 e^{2\gamma t} - 6\lambda\gamma^2 n e^{\gamma t}(\mu + 2ne^{\gamma t})}{(e^{\gamma t}+\mu)^2} + \frac{6k(8\pi+3\lambda)}{{a_0}^2\,(e^{\gamma t}+\mu)^{2n}}\right] \tag{14}$$

$$p(t) = \frac{1}{D}\left[\frac{3\gamma^2 n n e^{\gamma t}[2\,\lambda n\, e^{\gamma t} - (8\pi+3\lambda)(\mu+2ne^{\gamma t})]}{(e^{\gamma t}+\mu)^2} + \frac{3k\,(8\pi-3\lambda)}{{a_0}^2\,(e^{\gamma t}+\mu)^{2n}}\right] \tag{15}$$

$$\omega(t) = \left[\frac{\{3\gamma^2 n n e^{\gamma t}[2\,\lambda n\, e^{\gamma t} - (8\pi+3\lambda)(\mu+2ne^{\gamma t})]\}\,{a_0}^2\,(e^{\gamma t}+\mu)^{2n} + 3k\,(8\pi-3\lambda)(e^{\gamma t}+\mu)^2}{\{(8\pi+4\lambda)\,6\gamma^2 n^2 e^{2\gamma t} - 6\lambda\gamma^2 n e^{\gamma t}(\mu+2ne^{\gamma t})\}\,{a_0}^2\,(e^{\gamma t}+\mu)^{2n} + 6k(8\pi+3\lambda)(e^{\gamma t}+\mu)^2}\right] \tag{16}$$

where $D = (8\pi + 4\lambda)(8\pi + 3\lambda) - 2\lambda^2$.

Using equation (3) and (12), the Kaluza-Klein Friedmann-Robertson-Walker (FRW) type logamediate Hubble universe within the framework of *f(R,T)* gravity is

$$ds^2 = dt^2 - ({a_0}^2\,(e^{\gamma t} + \mu)^{2n})\left\{\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\varphi^2) + (1-kr^2)d\phi^2\right\} \tag{17}$$

3.1.1 Cosmological Dynamics of Model I:

Parameter	Early-Time Behaviour ($t \to 0^+$)	Late-Time Behaviour ($t \to 5$)	Interpretation
ρ(t)	High for $k = \pm 1$; lower for $k = 0$; all evolving toward convergence	Stabilizes to a nearly constant value	Dark energy dominance emerges; model supports different curvatures early on.
p(t)	Strongly negative, especially for $k = 0$; varies for $k = \pm 1$	Approaches small negative constant	Drives cosmic acceleration, curvature affects early dynamics.
ω(t)	From 0 (dust-like) or mild negative to strongly negative (near $\omega = -0.8$)	Converges to $\omega \approx -0.8$ for all k	Smooth transition from deceleration to acceleration; dark energy like behaviour.
H(t)	Steadily increases from(~0.7)	Asymptotically approaches (~1.2)	Intermediate expansion: slower than exponential but faster than power-law.
q(t)	Strongly negative ($q \sim -1.8$)	Approaches ($q \sim -1$)	Indicates sustained and strong acceleration throughout the evolution.
a(t)	Starts small and grows slowly	Grows very rapidly (super-exponential-like)	Continuous expansion, consistent with inflation and dark energy-driven growth.

The above table summarizes the cosmological dynamics of Model I, illustrating a smooth transition from an early, curvature-dependent phase to a stable, late-time accelerated universe. The model begins in a state of strong means super acceleration ($q \sim -1.8$), driven by a negative pressure. As the universe evolves, the energy density and pressure stabilize, and the equation of state parameter converges to a dark energy like value of $\omega \approx -0.8$. This dynamic causes the deceleration parameter to settle at ($q \sim -1$), leading to a sustained, rapid, super-exponential growth of the scale factor at late times. For small t means early times, $t \to 0$, $e^{\gamma t} \approx 1$ that implies $H(0) = \frac{\gamma n}{(1+\mu)}$, H is finite and positive (assuming γ, $n > 0$), the universe has an initial expansion rate dependent on γ, n and μ.[62] For large t means at late times, $e^{\gamma t} \gg \mu$, $H(t) \approx \gamma n$. Indicating that the Hubble parameter asymptotically approaches a constat value γn. This behaviour corresponds to a de Sitter like expansion, which is characteristic of dark energy-dominated universes with a cosmological constant.[16,18] For the deceleration parameter, at early times, $q_0 = -1 - \frac{\mu}{n}$, if μ, $n > 0$ then $q_0 < -1$ indicating a phantom like phase with super accelerated expansion where the expansion rate increases over time more rapidly than in a de Sitter universe.[33]

For large $q(t) \to -1 - \frac{\mu}{n e^{\gamma t}} \to -1$, the deceleration parameter tends to -1, confirming that the universe transitions toward a standard exponential acceleration, consistent with a cosmological constant scenario.[29] Thus, the universe starts in a super-accelerated (phantom) phase and asymptotically approaches a constant-acceleration phase with $q = -1$, showing a smooth transition from phantom-like behaviour to de Sitter expansion. The scale factor grows exponentially at late times. This is typical of late-time dark energy-dominated cosmology, where the accelerated expansion becomes dominant.[12,38] Overall, *a(t)* increases smoothly, with the curve becoming steeper over time and turning exponential at late times, confirming an accelerating cosmic expansion consistent with observations.

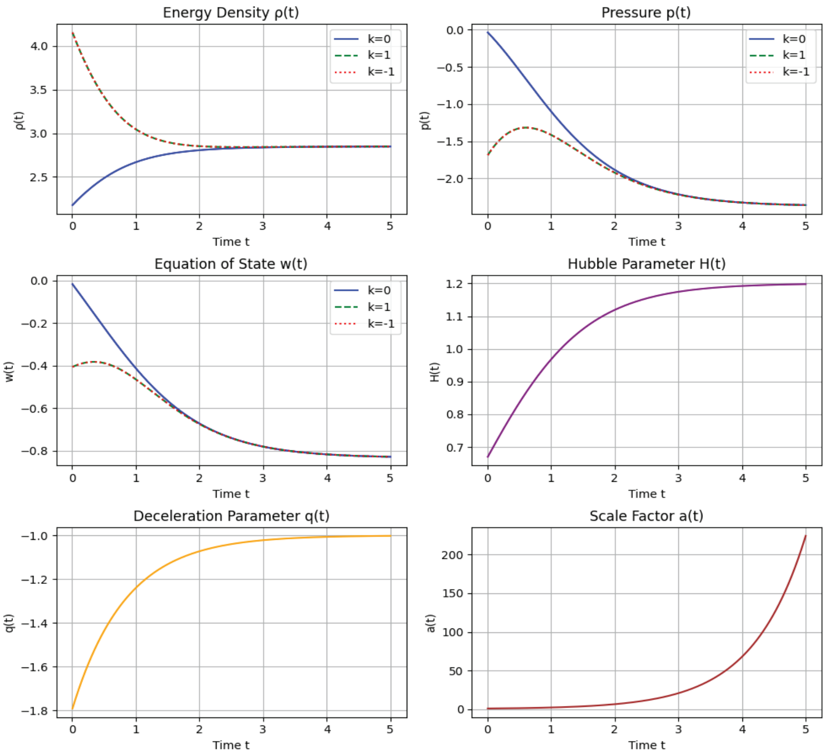

Fig.1. *Plots of cosmological parameters Vs. time for Model I*

Figure 1, provides a consolidated view of the Emergent exponential model (Model I), illustrating its evolution into a stable, accelerating universe. The energy density *(ρ)* remains positive while the pressure *(p)* becomes negative, with both parameters quickly evolving from curvature dependent initial values to a constant state at late times. This transition is clearly reflected in the equation of state *(ω)* which smoothly evolves from a dust-like or mild negative state and converges to $\omega \approx -0.8$, acting as a dark energy fluid. This dynamic is driven by a Hubble parameter *(H)* that steadily increases toward a constant value, causing a strong, sustained acceleration. This is confirmed by the deceleration parameter *(q)*, which begins in a super-accelerated phantom phase ($q \approx -1.8$) before rising to a standard de Sitter-like state ($q = -1$). Consequently, the scale factor *(a)* shows a rapid, exponential-like growth, confirming a continuously accelerating cosmic expansion.

3.2 Model II: Intermediate generalized power-law model

The time-dependent Hubble parameter

$$H(t) = \alpha\sigma t^{\sigma-1}. \tag{18}$$

represents a generalized power-law expansion model, which is widely used in cosmology to investigate various phases of the universe's evolution. In this expression α is a positive dimensional constant governing the scale of expansion, σ is a dimensionless index controlling the nature of expansion, and t is the cosmic time. The corresponding scale factor is given by $a(t) = e^{\alpha t^{\sigma}}$ which demonstrates accelerated or decelerated expansion depending on the choice of σ . For $\sigma = 1$, the model reduces to a de Sitter-like expansion $H(t) = \alpha$, producing an exponential scale factor $a(t) = e^{\alpha t}$, characteristic of a universe dominated by a cosmological constant or vacuum energy.[16] When $\sigma > 1$, the Hubble parameter increases with time, representing a super accelerated expansion or phantom energy behaviour, as discussed by Caldwell et al.[14] In contrast, for $0 < \sigma < 1$, *H(t)* decreases with time, corresponding to decelerated expansion, suitable for modelling early-time radiation or matter-dominated epochs.[15] These cases allow the universe to transition from deceleration to acceleration as supported by current observational data from supernovae and the cosmic microwave background.[18] The flexibility of this model lies in its capacity to describe both early- and late-time behaviours of the universe. Notably, for $\sigma > 1$, the scale factor remains finite at $t = 0$, offering a non-singular origin that supports bouncing or emergent cosmologies.[17] These models avoid the initial singularity problem of the standard Big Bang cosmology and align with quantum gravity expectations. Thus, this Hubble parametrization serves as a powerful phenomenological tool to explore the complete cosmic history from early deceleration to late time acceleration and to test various modified gravity and dark energy scenarios under observational constraints. For analysing the evolution and stability within the higher-dimensional Kaluza-Klein framework, one can easily derived the scale factor and deceleration parameter from equation (18) as follows;

$$a(t) = a_0 e^{\alpha t^{\sigma}}. \tag{19}$$

$$q(t) = -1 - \frac{1}{2t^2} \tag{20}$$

Furthermore, we have calculated the expressions for energy density, pressure and equation of state parameter with the use of equations

(18), (19) into (8)-(10) as follows;

$$\rho(t) = \frac{1}{D}\left[6\sigma\alpha\left((8\pi+4\lambda)\,t^{2(\sigma-1)} - \lambda(\sigma-1)t^{\sigma-2}\right) + 6k(8\pi+3\lambda){a_0}^{-2}e^{-2\alpha t^{\sigma}}\right] \tag{21}$$

$$p(t) = \frac{1}{D}\left[-3\sigma\alpha\left\{2\sigma\alpha\,(8\pi+2\lambda)t^{2(\sigma-1)} + (8\pi+3\lambda)(\sigma-1)t^{\sigma-2}\right\} - 3k(8\pi+\lambda){a_0}^{-2}e^{-2\alpha t^{\sigma}}\right] \tag{22}$$

$$\omega(t) = \left[\frac{-3\sigma\alpha\left\{2\sigma\alpha\,(8\pi+2\lambda)t^{2(\sigma-1)}+(8\pi+3\lambda)(\sigma-1)t^{\sigma-2}\right\}-3k(8\pi+\lambda){a_0}^{-2}e^{-2\alpha t^{\sigma}}}{6\sigma\alpha\left((8\pi+4\lambda)\,t^{2(\sigma-1)}-\lambda(\sigma-1)t^{\sigma-2}\right)+6k(8\pi+3\lambda){a_0}^{-2}e^{-2\alpha t^{\sigma}}}\right] \tag{23}$$

Using equation (3) and (19), the Kaluza-Klein Friedmann-Robertson-Walker (FRW) type logamediate Hubble universe within the framework of *f(R,T)* gravity is

$$ds^2 = dt^2 - \left({a_0}^2 e^{2\alpha t^{\sigma}}\right)\left\{\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + sin^2\theta d\varphi^2) + (1-kr^2)d\phi^2\right\} \tag{24}$$

3.2.1 Cosmological Dynamics of Model II:

The following table summarizes the cosmological dynamics of Model II, showing a realistic transition from a hot, dense early universe to a late-time accelerating phase. It begins with a high energy density and large positive pressure, corresponding to a strong deceleration period (positive q). As the universe evolves, the pressure becomes negative, causing the deceleration parameter to drop below zero, marking the onset of cosmic acceleration. The model stabilizes with an equation of state converging to ($\omega \approx 0.7$) which is consistent with quintessence-like dark energy and drives the rapid, continuous expansion of the scale factor.

Parameter	Early-Time Behaviour $t \to 0^+$	Late-Time Behaviour $t \to 5$	Remarks
$\rho(t)$	Diverges to high values	Stabilizes to a nearly constant value.	Reflects hot, dense beginning and approach to dark energy dominance.
$p(t)$	Large and positive	Small and negative.	Transition from matter radiation dominated era to accelerated phase.
$\omega(t)$	Positive or fluctuating, depending on k	Converges to $\omega = 0.7$.	Consistent with quintessence-like dark energy.
$H(t)$	Low initially, then rises $\sigma > 1$	Smooth and gradual increase.	Represents intermediate expansion between pow-er-law and exponential.
$q(t)$	Highly positive (deceleration)	Drops below zero (acceleration).	Signifies realistic cosmic transition to accelerated expansion.
$a(t)$	Starts small, grows slowly	Increases rapidly.	Continuous and accelerating growth of the Universe.

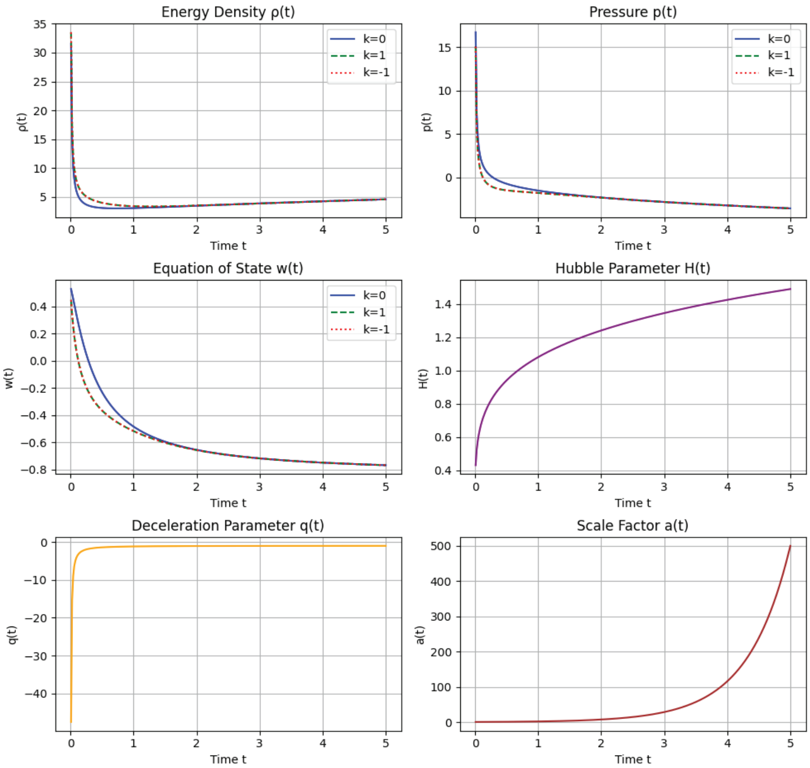

Fig.2 *Plots of cosmological parameters Vs. time for Model II*

From figure 2, it has been observed that for this model, the energy density starts extremely high and gradually stabilizes, reflecting a transition from a dense early universe to a dark energy dominated phase.[18] The pressure evolves from large positive values to small negative values, driving late-time acceleration.[49] The equation of state parameter *ω(t)* shifts from radiation or matter-like behaviour to values close to −1, mimicking dark energy.[16,37] The Hubble parameter increases with time but grows slower than in exponential expansion, indicating intermediate expansion scenarios.[62] The deceleration parameter begins positive (decelerating universe) and drops below zero, signalling a transition to acceleration. Meanwhile, the scale factor grows slowly at first and then rapidly increases, confirming continuous and accelerated cosmic expansion.[12]

3.3. Model III: Logamediate Model

The Hubble parameter

$$H(t) = \frac{\delta\beta\, log^{\beta-1} t}{t} \quad ,\delta > 0 \text{ and } \beta > 1. \tag{25}$$

represents a logarithmic time-dependent expansion rate that captures both early and late-time dynamics of the universe. This form is particularly suitable for modelling a smooth transition from early time inflation to a late time accelerated phase, making it valuable in unified cosmological scenarios. Feature of this parametrization is that it diverges near $t \to 1$, mimicking an inflationary burst, while at late times it decays slowly, effectively reproducing a dark energy-like behaviour with the equation of state approaching $\omega \approx$ -1. The logarithmic dependence introduces mild singularity-free behaviour and avoids the classical big bang singularity by initiating the universe from a finite *a(t)*. Such Hubble flow models have been widely studied to bridge the gap between early and late-time cosmic acceleration, consistent with observational datasets including SNe Ia, CMB, and BAO.[29,34]

For analysing the evolution and stability within the higher dimensional Kaluza-Klein framework, one can easily derived the scale factor and deceleration parameter from equation (25) as follows;

$$a(t) = e^{\delta(log t)^{\beta}} \quad ,\delta > 0 \text{ and } \beta > 1. \tag{26}$$

$$q(t) = -1 - \left[\frac{(\beta-1) - log t}{\delta\beta log t}\right]. \tag{27}$$

Furthermore, we have calculated the expressions for energy density, pressure and equation of state parameter with the use of equations (25), (26) into (8)-(10) as follows;

$$\rho(t) = \frac{1}{D}\left[\frac{6\delta\beta}{t^2}\left((8\pi + 2\lambda)\,\delta\beta(log t)^{2(\beta-1)} - \lambda\,(log t)^{\beta^2-4\beta+3}\right) + 6ke^{-2\delta(log t)^{\beta}}(8\pi + 3\lambda)\right], \tag{28}$$

$$p(t) = \frac{1}{D}\left\{\frac{3\delta\beta}{t^2}\left[-(8\pi + 3\lambda)\,(log t)^{\beta^2-4\beta+3} - 2(8\pi + 2\lambda)\delta\beta\,\,(log t)^{2(\beta-1)}\right] - 3(8\pi + \lambda)ke^{-2\delta(log t)^{\beta}}\right\}, \tag{29}$$

$$\omega(t) = \frac{1}{2}\left[\frac{\left(-(8\pi+3\lambda)\,(log t)^{\beta^2-4\beta+3} - 2(8\pi+2\lambda)\delta\beta\,\,(log t)^{2(\beta-1)}\right) - t^2(8\pi+\lambda)ke^{-2\delta(log t)^{\beta}}}{\left((8\pi+2\lambda)\,\delta\beta(log t)^{2(\beta-1)} - \lambda\,(log t)^{\beta^2-4\beta+3}\right) + t^2ke^{-2\delta(log t)^{\beta}}(8\pi+3\lambda)}\right], \tag{30}$$

Using equation (3) and (26), the Kaluza-Klein Friedmann Robertson Walker (FRW) type logamediate Hubble universe within the framework of *f(R,T)* gravity is,

$$ds^2 = dt^2 - \left(e^{2[\delta(log t)^{\beta}}\right)\left\{\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + sin^2\theta d\varphi^2) + (1-kr^2)d\phi^2\right\}. \tag{31}$$

3.3.1 Cosmological Dynamics of Model III:

The following table summarizes the dynamics of Model III, which describes a universe undergoing an early inflationary epoch followed by a stable, late-time accelerated expansion. Initially, the model experiences strong acceleration (deeply negative q) driven by a highly negative pressure, reflecting an inflationary phase. During this time, the equation of state (ω)is unstable and fluctuating. As the universe evolves, the Hubble parameter peaks and then gradually decreases, and all parameters stabilize. At late times, the energy density approaches a constant, the pressure remains negative, and ω converges to -1, mimicking dark energy and ensuring a sustained cosmic acceleration.

Parameter	**Early-Time Behaviour ($t \to 1^+$)**	**Late-Time Behaviour ($t \to 5$)**	**Remark**
ρ(t)	Very high (especially for closed universe), diverges as ($t \to 1$)	Decreases rapidly and stabilizes near a constant value	Reflects hot, dense initial state; stabilizes to a cosmological constant-like behaviour later

p(t)	Highly negative for (k = 0)indicating strong inflation; varies with curvature	Approaches a small negative constant, signifying late-time acceleration	Negative pressure drives early inflation and sustained late accelerated expansion
ω(t)	Strong fluctuations and divergences for (k = 0); smooth but high for (k = ±1)	Converges toward ($\omega \approx$ -1), mimicking dark energy	Transition from unstable early phase to stable dark energy-dominated epoch
H(t)	Increases to a peak value, modelling an initial inflationary phase	Gradually decreases, indicating slowing expansion rate	Captures the expected inflation peak and subsequent slower expansion
q(t)	Deeply negative (strong acceleration), especially for flat case	Less negative but still ($q < 0$), indicating sustained acceleration	Universe undergoes early rapid acceleration followed by milder late-time acceleration
a(t)	Starts from a finite value and grows gradually	Increases smoothly, confirming continuous expansion of the Universe	Consistent with non-singular start and continuous cosmic expansion

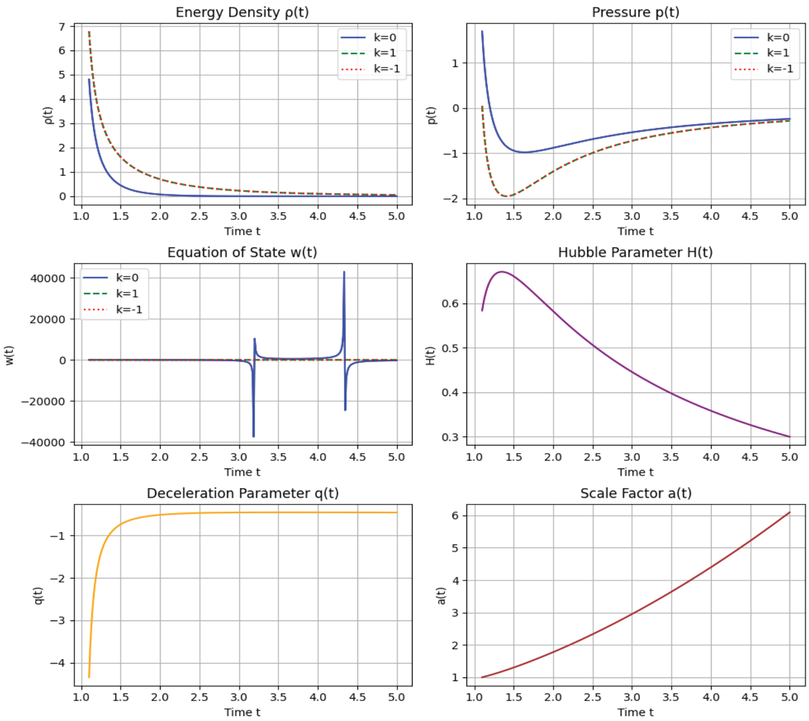

Fig.3: *Cosmological parameters plots Vs time for Model III*

From figure 3, it is observed that logarithmic Hubble parameter describes a logamediate expansion, characterized by a growth rate that is slower than exponential (de Sitter) yet faster than power-law expansion. The model starts with a small value of *H(t)*, increases gradually, peaks, and then slows down reflecting a physically meaningful expansion history. The corresponding scale factor increases steadily but sub-exponentially, making this framework highly compatible with observed cosmic acceleration.[62,29] The deceleration parameter *q(t)* exhibits a crucial feature: it is initially positive, indicating a decelerated phase, but transitions to negative values, signalling the onset of cosmic acceleration. This aligns well with current observational evidence that our universe underwent a transition from deceleration to acceleration in its recent past. From a theoretical standpoint, this form of *H(t)* offers several advantages: It supports a singularity-free origin (depending on parameters δ and β avoiding the classical Big Bang singularity. It provides a natural mechanism to drive late-time acceleration without invoking exotic fields. For specific values, such as $\delta = 2.0$ and $\beta = 2.55$, it mimics the behaviour of dark energy, making it viable within the framework of modified gravity and scalar field cosmology. Thus, the model captures a realistic and observationally consistent cosmic history from early deceleration to late-time acceleration while remaining analytically tractable and physically interpretable. This behaviour is consistent with an early inflation-like epoch followed by a late-time accelerated expansion in agreement with observations of cosmic evolution.

3.4. Model IV: Exponential decay model

This functional form

$$H(t) = H_0 - \frac{H_1}{e^{\xi t}} \quad (32)$$

has been explored in the context of late-time cosmology and dark energy reconstruction.[12,34] This time-dependent Hubble parameter, describes a smooth transition from an early decelerating phase to a late-time accelerating expansion. Initially, *H(t)* evolves rapidly due to the dominance of the exponential decay term $\frac{H_1}{e^{\xi t}}$ but it asymptotically approaches a constant value H_0, mimicking a de Sitter-like expansion. This parameter is ideal for unifying early and late cosmological behaviour, and consistent with supernovae and CMB observations.

For analysing the evolution and stability within the higher dimensional Kaluza-Klein framework, one can easily derived the scale factor and deceleration parameter from equation (32) as follows;

$$a(t) = a_0\, e^{\left(H_0\, t+\frac{H_1}{\xi e^{\xi t}}\right)} \tag{33}$$

$$q(t) = -1 - \frac{H_1 \xi e^{-\xi t}}{(H_0 - H_1\, e^{-\xi t})^2} \tag{34}$$

Furthermore, we have calculated the expressions for energy density, pressure and equation of state parameter with the use of equations (32), (33) into (8)-(10) as follows;

$$\rho(t) = \frac{1}{D}\left[(8\pi + 2\lambda)\left(6{H_0}^2 - 12\, H_0 H_1\, e^{-\xi t} + 6{H_1}^2 e^{-2\xi t}\right) - 6\lambda\xi H_1\, e^{-\xi t} + 6k\,(8\pi + 3\lambda){a_0}^{-2} e^{-2\, e^{\left(H_0\, t+\frac{H_1}{\xi e^{\xi t}}\right)}}\right] \tag{35}$$

$$p(t) = \frac{1}{D}\left[-(8\pi + 3\lambda)3\xi H_1\, e^{-\gamma t} - (8\pi + 2\lambda)(6{H_0}^2 - 12\, H_0 H_1\, e^{-\xi t} + 6{H_1}^2 e^{-2\xi t}) - (8\pi + \lambda)3k{a_0}^{-2} e^{-2\, e^{\left(H_0\, t+\frac{H_1}{\xi e^{\xi t}}\right)}}\right] \tag{36}$$

$$\omega(t) = \frac{1}{2}\left[\frac{-(8\pi+3\lambda)\xi H_1\, e^{-\xi t} - (8\pi+2\lambda)\left(2{H_0}^2 - 4\, H_0 H_1\, e^{-\xi t} + 2{H_1}^2 e^{-2\xi t}\right) - (8\pi+\lambda)k{a_0}^{-2} e^{-2\, e^{\left(H_0\, t+\frac{H_1}{\xi e^{\xi t}}\right)}}}{(8\pi+2\lambda)\left({H_0}^2 - 2\, H_0 H_1\, e^{-\xi t} + {H_1}^2 e^{-2\xi t}\right) - \lambda\xi H_1\, e^{-\xi t} + k\,(8\pi+3\lambda){a_0}^{-2} e^{-2\, e^{\left(H_0\, t+\frac{H_1}{\xi e^{\xi t}}\right)}}}\right] \tag{37}$$

Using equation (3) and (33), the Kaluza-Klein Friedmann-Robertson-Walker (FRW) type exponential decay Hubble universe within the framework of f (R,T) gravity is

$$ds^2 = dt^2 - \left({a_0}^2 e^{2\left(H_0\, t+\frac{H_1}{\xi e^{\xi t}}\right)}\right)\left\{\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + sin^2\theta d\varphi^2) + (1-kr^2)d\phi^2\right\} \tag{38}$$

3.4.1 Cosmological Dynamics of Model IV:

The cosmological model driven by $H(t) = H_0 - \frac{H_1}{e^{\xi t}}$ demonstrates a smooth transition from curvature-sensitive early dynamics to a universal accelerated expansion. Late-time convergence across all models supports a de Sitter like future, consistent with current observations of the universe's accelerated expansion. The following table describes the dynamics of Model IV, which undergoes a clear transition from a decelerating, curvature-dependent early phase to a stable, universal accelerated expansion. The model begins with strong deceleration *(q > 0)*, but as it evolves, the pressure becomes negative *(p ≈ -3)*, and the Hubble parameter rises to a constant value *(H_0)*. Consequently, the deceleration parameter asymptotes to *(q ≈ -1)*, confirming a late-time de Sitter-like phase where the scale factor grows exponentially, regardless of the initial spatial curvature.

Parameter	Early-Time Behaviour	Late-Time Behaviour	Remarks
ρ(t)	Varies by curvature; open universe starts highest.	All converge to *ρ ≈ 2*	Universe approaches constant energy density.
p(t)	Positive near zero initially for closed and open universes.	All settle to *p ≈ -3*	Indicates transition to dark energy-like negative pressure.
ω(t)	Highly sensitive initially; spikes in open model.	Converges to *ω ≈ 2*	Behaves like cosmological constant in the far future.
a(t)	Slow rise early on; closed model grows fastest.	Exponential-like increase	Reflects accelerated cosmic expansion.
q(t)	Strong deceleration or diverging behaviour at early times.	All asymptote to *q ≈ -1*	Confirms transition to accelerated expansion (de Sitter phase).
H(t)	Rapid rise from low values.	Stabilizes to constant H_0	Indicates approach to steady-state expansion.

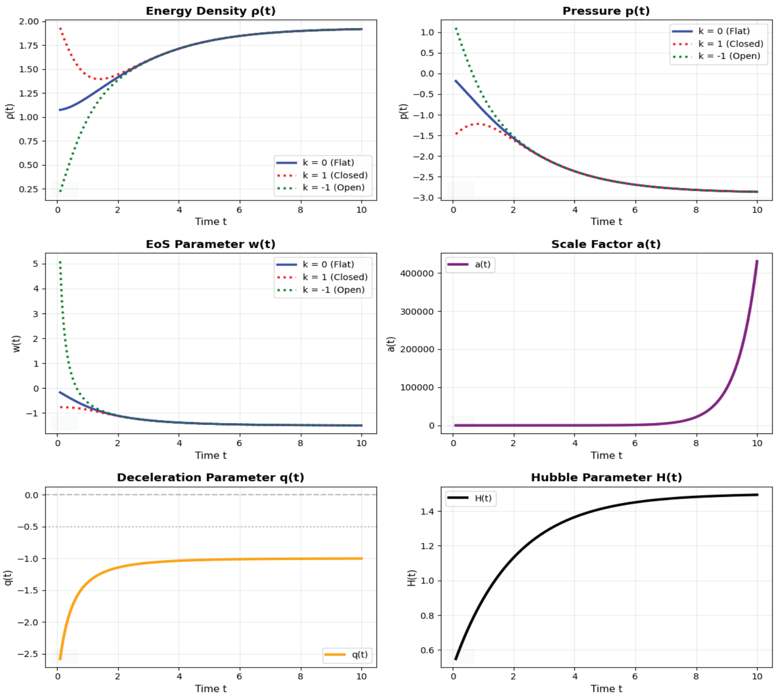

Fig.4: *Cosmological parameters plots Vs time for Model IV*

From figure 4, it is observed that, the plot of $\rho(t)$ typically monotonically decreases with cosmic time t, which is a physically realistic behavior. The decay rate depends on the logarithmic term, allowing for a smooth early dominance (high energy) and decay into a lower-energy late-time phase, consistent with observations of a transitioning universe.[15] The pressure $p(t)$ may start positive (decelerating era) and transition to negative values, indicating late-time acceleration driven by a dark energy component. The logarithmic term's behaviour can generate such a dynamic EoS fluid. This is analogous to scalar field models where pressure evolves from stiff matter to vacuum-dominated phases.[7] In the case of this logarithmic Hubble model, the plot typically shows that $q(t)$ evolves from positive to negative values, indicating a transition from a decelerating universe to an accelerating one. This is compatible with the observational discovery of cosmic acceleration using Type Ia supernovae data.[16] The transition behaviour also supports the idea of dynamical dark energy scenarios. For this model, $a(t)$ indicating a faster-than-power-law expansion at late times, which mimics accelerated cosmic expansion. This type of behaviour is similar to inflationary models and late-time dark energy dominance, where the expansion accelerates without bound.[36] In this exponential decay Hubble model, the evolution of $\omega(t)$ shows a transition from dust-like or stiff matter behavior to quintessence or phantom-like behaviour $\omega < -1$, depending on the values of δ and γ. This is consistent with phantom dark energy models that drive the universe into super-accelerated expansion phases.[33]

3.5. Model V: Emergent tanh expansion model

The Hubble parameter is taken as a hyperbolic tangent function

$$H(t) = \psi \tanh\left(\frac{t}{t_0}\right) \tag{39}$$

which captures a smooth transition from an early-time quasi-static or decelerated phase to a late-time accelerated expansion, consistent with emergent universe scenarios and avoiding the initial singularity problem.[19-22] This form allows the universe to begin from a static or nearly-static state, evolve through a decelerated expansion era, and asymptotically approach a de Sitter phase in the far future, without encountering a Big Bang-type singularity. The scale factor associated with this Hubble parameter reflects a bouncing-like or emergent behaviour, where the universe eternally exists with a finite size and gradually enters into a phase of accelerated expansion. In the framework of modified gravity, particularly within $f(R,T)$ gravity, where R is the Ricci scalar and T is the trace of the energy-momentum tensor, this choice of $H(t)$ becomes highly relevant. By incorporating the coupling between curvature and matter content, the model exhibits a rich phenomenology that can effectively explain cosmic acceleration, dark energy behaviour, and late-time modifications to General Relativity without relying on exotic scalar fields or fine-tuned cosmological constants.[1,23,24]

Such parametrizations have also been shown to yield viable cosmological models that satisfy observational constraints from Type Ia supernovae, baryon acoustic oscillations, and the cosmic microwave background.[25] Moreover, the hyperbolic tangent form introduces a natural time scale t_0 which governs the onset of acceleration and offers a framework to study transitions in the equation of state parameter and cosmographic diagnostics in a unified way.[26]

For analysing the evolution and stability within the higher-dimensional Kaluza-Klein framework, one can easily derived the scale factor and deceleration parameter from equation (39) as follows;

$$a(t) = a_0 \cosh^{\psi}\left(\frac{t}{t_0}\right) \quad . \tag{40}$$

$$q(t) = -1 - \frac{1}{\psi t_0 sinh^2\left(\frac{t}{t_0}\right)} \tag{41}$$

Furthermore, we have calculated the expressions for energy density, pressure and equation of state parameter with the use of equations (39), (40) into (8)-(10) as follows

$$\rho(t) = \frac{1}{D}\left[\frac{-6\psi\lambda}{t_0}\text{sech}^2\left(\frac{t}{t_0}\right) + 6\psi^2(8\pi - 2\lambda)\tanh^2\left(\frac{t}{t_0}\right) + 6k(8\pi + 3\lambda){a_0}^{-2}\,\text{sech}^{2\psi}\left(\frac{t}{t_0}\right)\right]. \tag{42}$$

$$p(t) = \frac{1}{D}\left[-(8\pi + 3\lambda)\frac{3\psi}{t_0}\text{sech}^2\left(\frac{t}{t_0}\right) - 6\psi^2(8\pi + 2\lambda)\tanh^2\left(\frac{t}{t_0}\right) - 3k(8\pi + \lambda){a_0}^{-2}\,\text{sech}^{2\psi}\left(\frac{t}{t_0}\right)\right]. \tag{43}$$

$$\omega(t) = \frac{1}{2}\left[\frac{-\psi(8\pi+3\lambda)\,\text{sech}^2\left(\frac{t}{t_0}\right) - 2t_0\psi^2(8\pi+2\lambda)\tanh^2\left(\frac{t}{t_0}\right) - kt_0(8\pi+\lambda){a_0}^{-2}\,\text{sech}^{2\psi}\left(\frac{t}{t_0}\right)}{-\psi\lambda\,\text{sech}^2\left(\frac{t}{t_0}\right) + t_0\psi^2(8\pi-2\lambda)\tanh^2\left(\frac{t}{t_0}\right) + kt_0(8\pi+3\lambda){a_0}^{-2}\,\text{sech}^{2\psi}\left(\frac{t}{t_0}\right)}\right], \tag{44}$$

Using equation (3) and (40), the Kaluza-Klein Friedmann-Robertson-Walker (FRW) type emergent tanh Hubble universe within the framework of *f(R,T)* gravity is

$$ds^2 = dt^2 - {a_0}^2\cosh^{2\psi}\left(\frac{t}{t_0}\right)\left\{\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + sin^2\theta d\varphi^2) + (1-kr^2)d\phi^2\right\} \tag{45}$$

3.5.1 Cosmological Dynamics for Model V:

This model, which includes late-time acceleration, matter-radiation transition, and early-time inflation, is a realistic cosmic scenario shows strong behaviour under changes in curvature as well. These findings characterize a cosmos that smoothly shifts from deceleration to acceleration, making it appropriate for models such as quasi-de Sitter cosmology or emergent cosmology. The following table summarizes the dynamics of Model V, illustrating a universe that begins in a strongly accelerating, inflationary-like phase (strongly negative q). The model then smoothly transitions, with the Hubble parameter (H) and energy density (ρ) stabilizing to constant values. At late times, the pressure (p) and equation of state (ω) both converge to (-1) This behaviour confirms a transition from an early inflationary epoch into a sustained, late-time accelerating phase, consistent with a de Sitter or ΛCDM-like dark energy universe.

Parameters	Early-time behaviour	Late-time behaviour	Interpretation
$\rho(t))$	High, dynamic	Constant (~1)	Dark energy phase
$p(t)$	Decreases from high to -1	$p \to -1$	Dark energy pressure
$\omega(t)$	Diverse ($\omega > 1$ and $\omega < 0$)	$\omega \to -1$	Quintessence/ΛCDM
$a(t)$	Exponential growth	Continues to grow	Accelerating universe
$q(t)$	Strongly negative	Moderate negative	Inflation + acceleration
$H(t)$	Grows to ψ	Constant	Asymptotic de Sitter

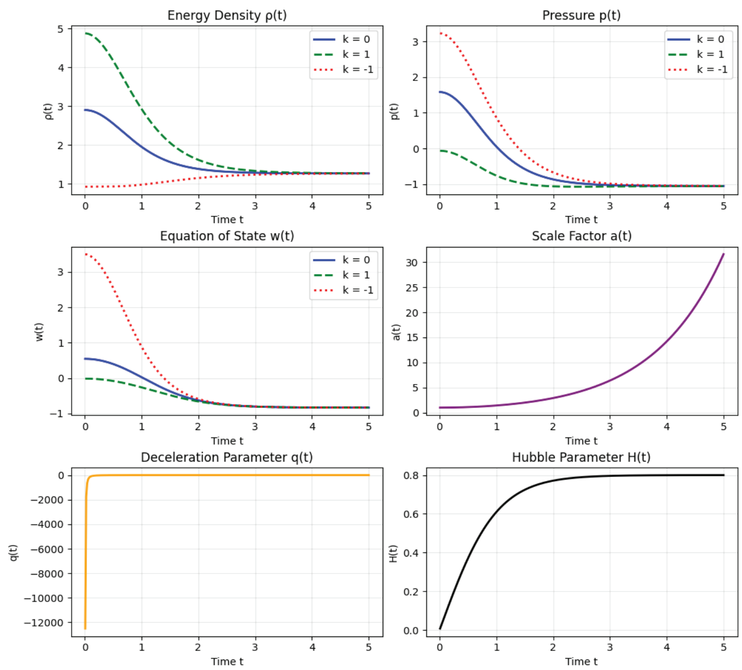

Fig. 5: *Plots of cosmological Parameters Vs, time for Model V*

From figure 5, it is observed that, the scale factor $a(t) = a_0 \cosh^{\psi}\left(\frac{t}{t_0}\right)$ which grows monotonically with time, indicating an expanding universe. The scale factor increases monotonically, showing an expanding universe. The growth becomes exponential-like at late times, consistent with accelerating expansion (where $\psi = 1.5$, $t_0 = 2.0$ and $a_0 = 1.0$). The plot displays the deceleration parameter $q(t)$, which starts from a large negative value and asymptotically approaches $q = -1$, characteristic of accelerated expansion similar to a de Sitter phase. Initially, $q(t) > 0$, indicating decelerated expansion. As time increases, q(t)becomes negative and approaches -1, reflecting a smooth transition to cosmic acceleration. The plot of the Hubble parameter $H(t) = \psi \tanh\left(\frac{t}{t_0}\right)$: It starts near 0 at early times and asymptotically approaches α as $t \to 0$ to $t \to \infty$ indicating a transition to a de Sitter-like accelerated expansion phase. This behaviour reflects a transition from a decelerated phase to a late-time accelerated (quasi-de Sitter) phase. The Hubble parameter starts near zero and increases with time.

4. Comparative analysis of Five Hubble models

Having analysed each model individually, we now compare their collective behaviour to understand their distinct cosmological implications. Figure 6 provides an overlay of all five models, while the following table summarizes their key evolutionary traits.

Model	Early Behaviour	Late-Time Behaviour	Physical Insight
Model I	Rapid decay in ρ, p ; modest acceleration.	Stabilizes to DE-like expansion.	Viable model with early inflation
Model II	Divergent $\omega(t)$, strong phantom traits.	Stabilizes, moderate expansion.	Points to an exotic early phase, possibly a bounce.
Model III	Smooth evolution in ρ,p and q.	Tracks ΛCDM behaviour.	Appears balanced and realistic kinematically.
Model IV	Explosive growth in $a(t)$, high $H(t)$	Super-accelerated.	Represents a phantom or inflation-like scenario.
Model V	Singular early behaviour, quick damping.	Approaches steady DE phase.	Shows transient bounce-like characteristics.

As the table and plots show, all models eventually support accelerated cosmic expansion, but their early time behaviours differ significantly. We now analyse the specific parameters from Figure 6 in detail.

4.1. Cosmological Analysis of Figure 6:

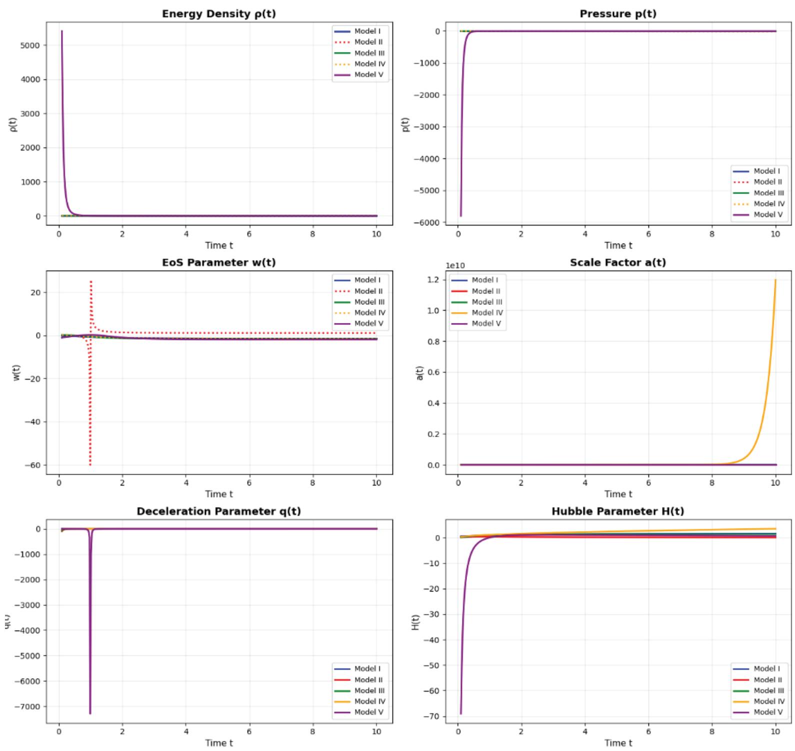

Fig.6: *Plots of cosmological Parameters of five Hubble Models Vs. time*

Energy Density $\rho(t)$ and Pressure $p(t)$: All models show energy density $\rho(t)$ starting at a very high value and decaying rapidly, converging to a small, positive constant at late times. This aligns with a hot, dense Big Bang-like origin followed by cosmic dilution and a late-time dark energy dominance.[32,40] The pressure $p(t)$ for all models diverges negatively at $t \to 0$ and asymptotically approaches a negative constant value.[33,42] This negative pressure is the necessary driver for the late-time accelerated expansion. **Equation of State ω(t):** The EoS parameter shows significant variation at early times, with Model II in particular exhibiting strong phantom behaviour *(ω ≈ -1)*.[29,14] At late times, all models converge toward *(ω ≈ -1)*, mimicking a cosmological constant. This indicates that, regardless of their early-time dynamics, all models evolve into a de Sitter-like dark energy-dominated phase. **Expansion Dynamics *a(t)*, *q(t)* and H(t):** The expansion dynamics confirm the dark energy behaviour. The scale factor *a(t)* shows an exponential-like growth for all models, with Model IV being the most rapid means super-acceleration. The deceleration parameter *q(t)* starts with large negative values suggesting an early inflationary phase and then stabilizes to a value consistent with sustained acceleration.[16,37] The Hubble parameter *H(t)* also shows rapid early-time variation before stabilizing, with Model IV (orange line) clearly maintaining the fastest expansion rate.[32] Kinematically, all five models successfully describe a universe that begins with a high-density state and evolves into a late-time accelerated expansion. However, their early-time dynamics are drastically different, suggesting distinct physical origins. This analysis of what can happen now sets the stage for our final tests in the following sections: is it stable in Section 5 and is it physically allowed in Section 6.

5. Comparative study of Stability of Five Hubble Models

In this section, we proceed with the stability analysis for our five cosmological models. For a fluid description of the universe, the model is stable against small perturbations if the squared speed of sound is positive $(c_s^2 > 0)$. A negative value $(c_s^2 < 0)$ would imply an imaginary sound speed, leading to Laplacian instabilities. Since our derived pressure and density are functions of time, *p(t)* and *ρ(t)*, the squared sound speed is calculated as: $c_s^2 = (\partial p/\partial \rho) = \dot{p}/\dot{\rho}$. The red dashed line at $c_s^2 = 0$ in our plots (Fig. 7) is the threshold between stable and unstable regimes. We will now analyse each model based on this criterion.

Model	Early-Time Stability	Late-Time Stability	Comments
I	Unstable	Unstable	Disfavoured
II	Unstable	Stable	Acceptable after early time
III	Slight Instability	Fully Stable	Most Viable
IV	Highly Unstable	Stable	Acceptable with early quantum phase
V	Unstable	Becomes unstable again	Viable in transient phase

From fig. 7, it is observed that, in Model I, the squared sound speed $c_s^2 < 0$ throughout cosmic evolution, rising from a highly unstable value near (-14) to around (-4), but never becoming positive. This indicates a classically unstable evolution, inconsistent with perturbation stability criteria in standard and modified gravity.[47,48] Hence, Model I is not viable for a perturbatively stable cosmology. For Model II, there is initial instability due to a sharp negative dip, followed by a large overshoot into a positive region. After $t \gtrsim 2, c_s^2$ stabilizes to small positive values. This behaviour reflects a conditionally stable model with early-time phantom-like instability.[33,51] potentially resolving into a viable dark energy model as the universe evolves. Model III exhibits only a brief early-time instability, followed by quick stabilization to a constant positive value $c_s^2 \approx 4.5$. This near-total positive behaviour aligns with physically acceptable cosmologies in both scalar field and modified gravity frameworks,[46] making Model III the most stable and viable among all. Model IV undergoes severe early-time instability, with $c_s^2 \lesssim -150$ suggesting a highly unstable regime near the initial singularity. However, it transitions into a stable phase with $c_s^2 \geq 0$ at later times. Such behaviour may be interpreted as reflecting an early quantum-dominated inflationary phase, followed by classical stabilization.[42,43] Model V starts with instability $c_s^2 < 0$ becomes briefly stable withi $0 < c_s^2 < 2$ during $2 \lesssim c_s^2 \lesssim 6$, then decays back into instability. This transient stability resembles models with brief classical phases embedded within quantum or phantom dynamics,[44,49] requiring further modification or additional fields for long-term viability.

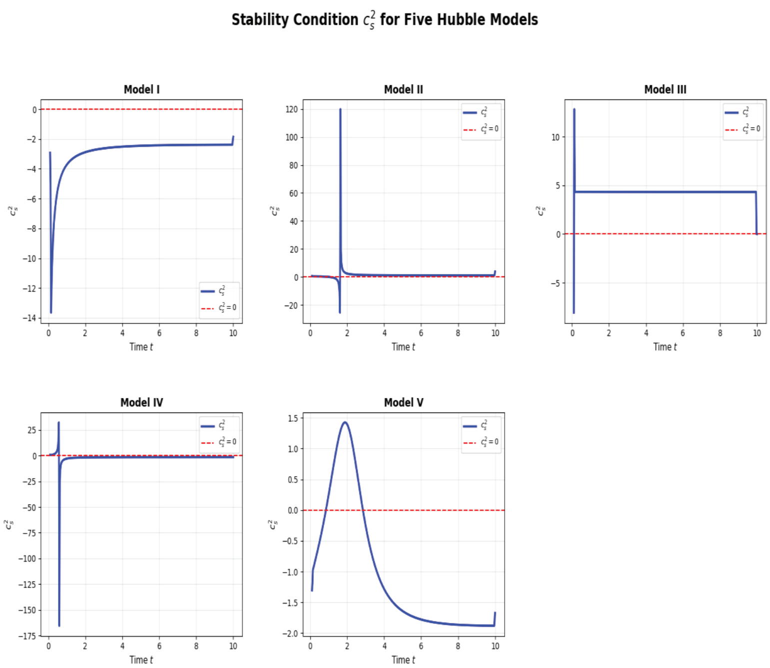

Fig. 7: *Plots of* c_s^2 *Vs. time for multiple Hubble Models*

From the perturbative stability perspective, Model III is the most reliable, exhibiting a consistently positive sound speed. Models II and IV are conditionally viable, provided early instabilities are attributed to quantum corrections or inflationary epochs. In contrast, Models I and V are predominantly unstable and are less favourable unless new stabilizing mechanisms are introduced.[50]

6. Comparative Analysis of Energy Conditions

Based on the energy conditions: WEC $(\rho \geq 0)$, NEC $(p + \rho \geq 0)$, SEC $(\rho + 3p \geq 0)$ and DEC $(\rho \pm p \geq 0)$, we have plotted all energy conditions for the five Hubble models. In this section, we interpret how well each one satisfies the physical viability criteria. It is critical to note that this analysis must be read in conjunction with the stability analysis from Section 5. A model must satisfy both perturbative stability $(c_s^2 > 0)$ and (at minimum) the WEC and NEC to be considered physically viable.

From Fig. 8, it is observed that in Model I becomes physically viable because all energy conditions are satisfied after a short initial phase. Early-time violations suggest an inflationary or exotic matter phase, transitioning smoothly to a standard matter-dominated era, which is a common feature in emergent universe and loop-inspired inflationary scenario.[19,20] Such models are known to violate the Strong Energy Condition (SEC) temporarily while maintaining physical viability at late times. However, as demonstrated in our stability analysis in section 5, this model is perturbatively unstable throughout its evolution $(c_s^2 < 0)$. Therefore, despite appearing to pass the energy condition test at late times, Model I is not physically viable due to this underlying instability. Model II is marginally viable. Violations of the SEC and brief early-time WEC/NEC violations may hint at a bouncing or phantom-like phase, a feature consistent with *f(R,T)* and *f(Q,T)* gravity theories.[24,27] The stability improves as the universe evolves, indicating that initial violations could be attributed to effective quantum gravitational corrections or modified geometrical couplings.[28]

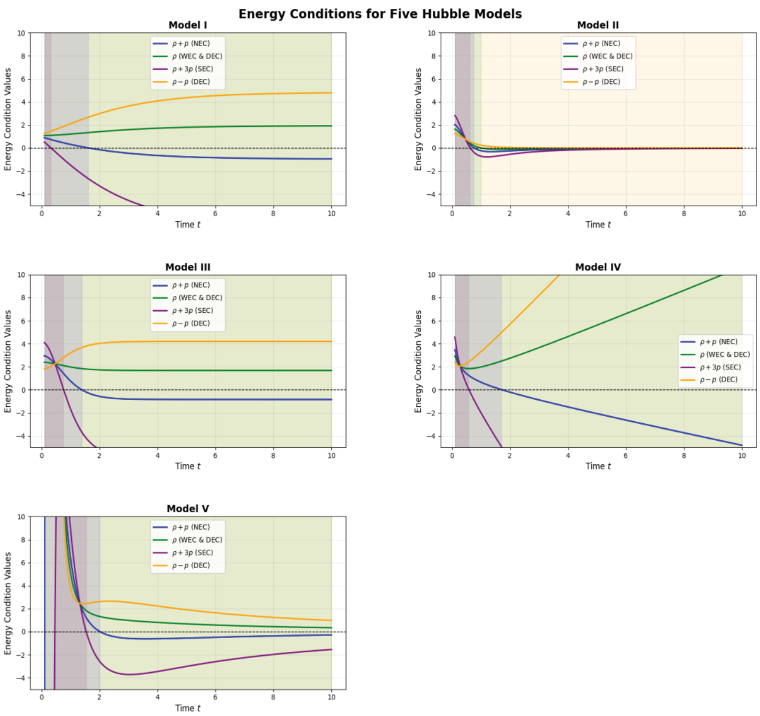

Fig.8: *Plots of Energy condition Values Vs. time for all Hubble Models*

Model III is realistic at later times, with violations confined to early-time regimes. This pattern suggests a smooth exit from an exotic matter or quantum-dominated era toward a viable classical universe, a behaviour also noted in loop quantum cosmology and non-singular bouncing scenarios.[42,44] **Model IV** is not physically acceptable under traditional ECs. Persistent violation of NEC and SEC indicates the dominance of exotic energy components, possibly mimicking phantom or super-inflationary behaviour. Similar conditions have been explored in the context of phantom dark energy and modified gravity where such violations lead to future singularities or non-standard evolution.[14,29] **Model V** behaves similarly to a universe with an early-time singularity or quantum gravity phase, like the Big Bounce, and becomes viable only in the classical regime. Transient EC violations are often considered tolerable in many modified gravity theories,[31,32] especially those designed to resolve the initial singularity problem while allowing a transition into standard cosmology.

7. Conclusions

This study has conducted a comprehensive theoretical investigation into the cosmological dynamics of a Kaluza-Klein FRW universe within the *f(R,T)* modified gravity framework. By employing a reconstruction method based on five distinct Hubble parameter parametrizations, we have explored a wide range of cosmic histories, including emergent, bouncing, and transitional scenarios. While most of the models successfully describe the universe's transition from an early-time (matter or radiation) phase to a late-time accelerated expansion driven by negative pressure, our analysis demonstrates that kinematic consistency alone is not a sufficient criterion for physical viability.

The crucial differentiator in our findings is the perturbative stability analysis. We found that:

- The Logamediate Model (Model III) is the most physically viable, exhibiting a brief early-time instability before quickly stabilizing to a positive squared sound speedfor ($c_s^2 > 0$) the rest of its evolution.
- The Power-law (Model II) and Exponential Decay (Model IV) models are only conditionally viable, as they suffer from severe early-time instabilities ($c_s^2 < 0$) even if they stabilize at late times.
- The Emergent Exponential (Model I) and Emergent Tanh (Model V) models are found to be theoretically disfavoured, as they remain unstable throughout their entire evolution.

Our work establishes that the stability analysis acts as a critical filter. Simply finding a model that matches the observed expansion history with *(q < 0)* is insufficient if the model is unstable to its own perturbations. Of the five models tested, only the Logamediate parametrization presents a dynamically stable and self-consistent cosmological history within this *f(R,T)* Kaluza-Klein framework.

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